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LETTER TO THE EDITOR

Planck distribution for a q-boson gas

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Abstract. The energy density distributions for bosons obeying the q-deformation of the harmonic oscillator algebra have been studied in order to obtain some physical insight into the parameter q. The action of this parameter on the energy distributions resembles the action of the temperature parameter in the 'classical' Planck law ($q = 1$).

Quantum deformation of Lie algebras and groups [1–4] has been shown to be deeply rooted in many problems of physical and mathematical interest, such as rational conformal field theories (RCFT) [5–8], exactly solvable statistical models [9], inverse scattering theory applied to integrable models in quantum field theories [10], non-commutative geometry [3], knot theory in three dimensions, etc. In all these disparate areas of mathematical physics the Yang–Baxter equation plays an essential role. More recently, Macfarlane [11] and Biedenharn [12, 13] have constructed a realization of the simplest quantum group $\mathcal{U}_q(\mathrm{SU}(2))$ using a q-analogue of the bosonic harmonic oscillator algebra. These q-oscillator techniques have been applied to quantum superalgebras in [14–16].

One of the most interesting issues is to study the physics behind the q-structures in order to get some insight into the physical implications of these deformations [17, 18]. In this letter we take a small step in this direction by studying the modifications in the energy density distributions of a boson gas when these particles obey a q-deformation of the canonical commutation relations.

The usual way to introduce the q-boson oscillators is through the Jordan–Schwinger construction of the classical algebra of $\mathrm{SU}(2)$ [11–13]‡.

The quantum Lie algebra (or quantum group) $\mathcal{U}_q(\mathrm{SU}(2))$ is a deformation of the universal enveloping algebra of $\mathrm{SU}(2)$ which is endowed with a Hopf algebra structure [20, 1, 7, 8]. The quantum algebra $\mathcal{U}_q(\mathrm{SU}(2))$ can be characterized by giving its three generators J_+, J_-, J_z together with the following defining relationships based on the Chevalley basis of $\mathrm{SU}(2)$:

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} \equiv [2J_z]_q \tag{2}$$

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‡ Recently, in [19] the q-deformed oscillator algebra is shown to be a quantum group itself with the supplied Hopf structure.

where q is the deformation parameter of the classical algebra $\mathrm{SU}(2)$. It is a real number or it has unit modulus in order to be compatible with the adjoint operation ($J_\pm^\dagger = J_\mp$) with the algebra structure (1),(2):

$$q \in \mathbb{R} \qquad q \equiv \mathrm{e}^{\tau} \tag{3a}$$

$$q \in S^1 \qquad q \equiv \mathrm{e}^{\mathrm{i}\theta}. \tag{3b}$$

As usual, it is convenient to introduce q-numbers denoted by $[x]_q$:

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \xrightarrow{q \to 1} x. \tag{4}$$

The algebra $\mathrm{SU}(2)$ is recovered from (1),(2) in the limit $q \to 1$.

The representation theory of the quantum group is quite similar to classical theory (when q is not a root of unity). Several authors [2, 21, 11] have proved that there exist irreps of $\mathcal{U}_q(\mathrm{SU}(2))$ labelled with $j = 0, \frac{1}{2}, 1, \ldots$ acting on a Hilbert space V^j with basis vectors

$$|jm\rangle_q \qquad -j \leqslant m \leqslant j \tag{5}$$

as follows

$$J_z |jm\rangle_q = m|jm\rangle_q \tag{6}$$

$$J_\pm |jm\rangle_q = \sqrt{[j \mp m]_q [j \pm m + 1]_q} |jm \pm 1\rangle_q. \tag{7}$$

From (7) we see that the usual numbers have turned into q-numbers. The irrep V^j has dimension $2j + 1$.

The q-boson oscillator realization of $\mathcal{U}_q(\mathrm{SU}(2))$ (equations (1), (2)) is given by the following Jordan–Schwinger map:

$$J_+ = a_{1q}^\dagger a_{2q} \qquad J_- = a_{2q}^\dagger a_{1q} \qquad 2J_z = a_{1q}^\dagger a_{1q} - a_{2q}^\dagger a_{2q} \tag{8}$$

where $a_{iq}, a_{iq}^\dagger$ $(i = 1, 2)$ are two commuting copies of q-boson harmonic oscillators verifying the following q-deformed commutation relationship [11, 12]:

$$a_q a_q^\dagger - q^{-1} a_q^\dagger a_q = q^{N_q} \tag{9}$$

where N_q is the Hermitian number operator defined by

$$a_q^\dagger a_q = [N_q]_q = \frac{q^{N_q} - q^{-N_q}}{q - q^{-1}} \xrightarrow{q \to 1} N. \tag{10}$$

Then the usual commutation relationships of N_q with $a_q^\dagger, a_q$ hold:

$$\begin{aligned} [N_q, a_q^\dagger] &= a_q^\dagger \\ [N_q, a_q] &= -a_q. \end{aligned} \tag{11}$$

The highest weight representations of the q-Heisenberg algebra (9) are readily built up. Let $|0\rangle_q$ be the vacuum state such that $a_q|0\rangle_q = 0$. Define the number states $|n\rangle_q$ as usual:

$$|n\rangle_q \equiv \frac{(a_q^\dagger)^n|0\rangle_q}{\sqrt{[n]_q!}} \qquad \text{with} \qquad [n]_q! \equiv [n]_q[n-1]_q \ldots [1]_q. \tag{12}$$

Then the irreps of the q-oscillator are

$$N_q|n\rangle_q = n|n\rangle_q \tag{13}$$

$$a_q^\dagger|n\rangle_q = \sqrt{[n+1]_q}\ |n+1\rangle_q \tag{14}$$

$$a_q|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q. \tag{15}$$

Let us note that the action of N_q (13) is not q-deformed. The irreps (5) of $\mathcal{U}_q(\mathrm{SU}(2))$ are built up from the irreps (12) as tensorial products:

$$|jm\rangle_q = |j+m\rangle_q \otimes |j-m\rangle_q. \tag{16}$$

Once the Hilbert space of states $|n\rangle_q$ is constructed we can go further and reproduce many of the usual computations in ordinary quantum mechanics in order to gain some insight into the effect produced by the q-deformation of the Heisenberg algebra and its representations. So, we can look for a q-Heisenberg principle [12, 13]. Then, let us introduce the position X_q and momentum P_q operators as follows:

$$X_q \equiv \sqrt{\frac{\hbar}{2m\omega}}(a_q^\dagger + a_q) \tag{17}$$

$$P_q \equiv \mathrm{i}\sqrt{\frac{m\hbar\omega}{2}}(a_q^\dagger - a_q). \tag{18}$$

Let $\Delta_n X_q$ and $\Delta_n P_q$ be the uncertainties (quadratic mean deviations) of these operators in the states $|n\rangle_q$, then [22]

$$\Delta_n X_q \Delta_n P_q \geqslant \tfrac{1}{2}|_q\langle n|[X_q, P_q]|n\rangle_q|. \tag{19}$$

As $\mathrm{i}[P_q, X_q] = [a_q, a_q^\dagger] = \hbar([N_q+1]_q - [N_q])_q)$ the following q-uncertainty principle holds

$$\Delta_n X_q \Delta_n P_q \geqslant \frac{\hbar}{2}\frac{\cosh((n+\frac{1}{2})\ln q)}{\cosh(\frac{1}{2}\ln q)} \equiv \frac{\hbar(n,q)}{2} \xrightarrow{q\to 1} \frac{\hbar}{2}. \tag{20}$$

Here $\hbar(n,q)$ denotes an effective Planck constant that grows with n ($q \neq 1$), i.e. it is an uncertainty principle with a constant $\hbar(n,q)$ that is modified for each level n of the energy. Roughly speaking, it is as if the minimum uncertainty cell in the phase space increases as the energy increases.

Another way to obtain an insight into the physical implications of q-deforming the harmonic oscillator Hilbert space of states is to study its energy spectrum $E_n(q)$ and energy density distribution $U_q(\omega, T)$ associated with a q-boson gas (a q-black-body or a 'q-sun').

Let us introduce the q-oscillator Hamiltonian H_q [12]:

$$H_q = \frac{P_q^2}{2m} + \frac{m\omega^2}{2} X_q^2 \tag{21}$$

which can also be written in terms of the creation/annihilation operators or number operators:

$$H_q = \tfrac{1}{2}\hbar\omega(a_q^\dagger a_q + a_q a_q^\dagger) = \tfrac{1}{2}\hbar\omega([N_q]_q + [N_q + 1]_q). \tag{22}$$

This Hamiltonian is diagonal in the base $|n\rangle_q$ and its eigenvalues $E_n^0(q)$ are

$$E_n^0(q) = \tfrac{1}{2}\hbar\omega([n]_q + [n+1]_q) \xrightarrow{q\to 1} \hbar\omega(n + \tfrac{1}{2}). \tag{23}$$

The superindex in $E_n^0(q)$ means that the contribution from the zero-point energy of the oscillator has been taken into account. From (22) it follows that the energy levels are no longer uniformly spaced when $q \neq 1$.

In order to study and make graphical representations of the energy densities $U_q(\omega, T)$ it is convenient, as usual, to redefine the origin of energies from the zero point energy onward in such a way that

$$E_n(q) \equiv \tfrac{1}{2}\hbar\omega([n]_q + [n+1]_q - 1) \xrightarrow{q\to 1} \hbar\omega n. \tag{24}$$

Now we introduce the partition function $Z_q(\omega, T)$ for a q-boson gas in the canonical ensemble:

$$Z_q(\omega, T) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n(q)} \qquad \beta \equiv \frac{1}{k_B T}. \tag{25}$$

Then we can define the average energy density for this gas in the usual way:

$$U_q(\omega, T) \equiv \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2 \left[-\frac{\partial}{\partial\beta} \ln Z_q(\omega, T)\right] = \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2 \frac{\sum_{n=0}^{\infty} E_n(q) \mathrm{e}^{-\beta E_n(q)}}{\sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n(q)}}. \tag{26}$$

In the limit $q \to 1$ the 'classical' Planck law is recovered:

$$U_1(\omega, T) = \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2 k_B T \frac{\beta\hbar\omega}{\mathrm{e}^{\beta\hbar\omega} - 1}. \tag{27}$$

In order to study the energy density (25) when $q \neq 1$ it is convenient to use the following adimensional quantities:

$$x \equiv \frac{\hbar\omega}{k_B T} \tag{28}$$

$$z_q(x) \equiv \sum_{n=0}^{\infty} \exp\left\{-\tfrac{1}{2}x([n+1]_q + [n]_q - 1)\right\} \tag{29}$$

$$\begin{aligned} u_q(x) &\equiv x^3 \left[-\frac{\mathrm{d}}{\mathrm{d}x} \ln z_q(x)\right] \\ &= x^3 \frac{\sum_{n=0}^{\infty} \left[\frac{1}{2}([n+1]_q + [n]_q - 1)\right] \exp\left\{-\frac{1}{2}x([n+1]_q + [n]_q - 1)\right\}}{\sum_{n=0}^{\infty} \exp\left\{-\frac{1}{2}x([n+1]_q + [n]_q - 1)\right\}}. \end{aligned} \tag{30}$$

We notice that these quantities are invariant under the change $q \to q^{-1}$.

The convergence of the series $z_q(x)$ can be analysed as follows. When $q = \mathrm{e}^{\mathrm{i}\theta}$ is of modulus unity (3*b*) the energy spectrum oscillates with n, namely

$$E_n(\theta) = \frac{\hbar\omega}{2}\left[\frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)} - 1\right] \tag{31}$$

so that the general term of the series (28) does not tend to zero when $n \to \infty$, and $z_q(x)$ diverges in this case. The same holds when q is real and negative.

From now on we will concentrate on the case when q is real and positive i.e. $q = \mathrm{e}^{\tau}$ (3*a*). In this case the convergence of the series $z_q(x)$ can be determined using the following property of q-numbers:

$$[n]_q \geqslant n \qquad \forall n, q = \mathrm{e}^{\tau} \tag{32}$$

then $\exp\left\{-x\left[\frac{1}{2}([n+1]_q + [n]_q - 1)\right]\right\} \leqslant \mathrm{e}^{-xn}$ from which the following bound for the q-boson partition function in terms of the classical partition function ($q = 1$) holds:

$$z_q(x) \leqslant z_1(x) = 1/(1-\mathrm{e}^{-x}). \tag{33}$$

We proceed now to study the regime of great energies and/or low temperatures, i.e. $x = \hbar\omega/k_{\mathrm{B}}T \gg 1$. Then we can approximate the series in $z_q(x)$ by the first two terms:

$$z_q(x) \simeq 1 + \exp\left\{-\tfrac{1}{2}x([2]_q + [1]_q - 1)\right\} = 1 + \exp\left\{-\tfrac{1}{2}x([2]_q)\right\} \equiv z_q^{\mathrm{W}}(x) \tag{34}$$

and when inserted in (30), within this approximation, we obtain the q-extension of Wien law for the energy density $u_q^{\mathrm{W}}(x)$:

$$u_q^{\mathrm{W}}(x) \simeq x^3 \cosh\tau\,\mathrm{e}^{-x\cosh\tau} \tag{35}$$

where we have used $[2]_q = \frac{1}{2}(q + q^{-1}) = \cosh\tau$. Notice that the limit $q \to 1$ (or $\tau \to 0$) again reproduces the classical result of the Wien law.

Figure 1 illustrates the energy density $u_q^{\mathrm{W}}(x)$ for several values of the deformation parameter. Although expression (32) is only valid when $x \gg 1$, we have drawn the curves for all $x \in [0, \infty)$ as usual. In this way we see how the maximum x_{max} of the energy distribution decreases as q grows (or decreases as well, due to the symmetry $q \to q^{-1}$ in the distributions). In fact, it is easy to obtain the following extension of the Wien shift law:

$$x_{\mathrm{max}} = 3/\cosh\tau \tag{36}$$

or equivalently

$$\omega_{\mathrm{max}} = \frac{3}{\hbar\cosh\tau}k_{\mathrm{B}}T. \tag{37}$$

Let us notice that *the effect on ω_{max} when the parameter q grows is opposed to the one produced by an increment in temperature, but they both produce an analogous effect when they decrease.*

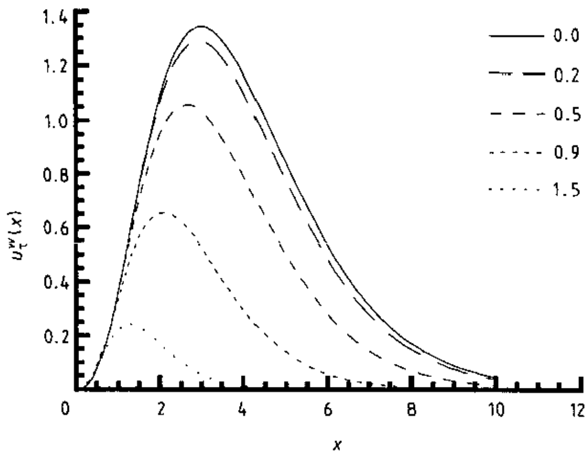

Figure 1. Energy density curves $u_\tau^{\mathrm{W}}(x)$ (35) in the Wien approximation for different values of the deformation parameter $\tau = \ln q$.

It is also worthwhile to notice in figure 1 how *the total energy* u_τ^{W} *(area) of the* q*-boson gas decreases as the deformation parameter* $q = \mathrm{e}^\tau$ *increases (or decreases as well, due to the symmetry* $q \to q^{-1}$*).* In fact, the q-Stefan law can be readily obtained in the Wien approximation by integrating (34), namely

$$U_\tau^{\mathrm{W}}(T) = \frac{6k_{\mathrm{B}}^4}{\pi^2(\hbar\cosh\tau c)^3}T^4 \tag{38}$$

and the decrease in the Stefan–Boltzmann constant with $q(\tau)$ is manifest.

From (36) and (37) we see that it is possible to understand the effect of the deformation in this approximation by means of an effective Planck constant $\hbar_\tau$:

$$\hbar_\tau = \hbar\cosh\tau \tag{39}$$

which reduces to $\hbar$ when $\tau \to 0$.

The analysis of the q-Planck law (25) for any value of x is more cumbersome due to the nature of the energy levels $E_n(q)$ (23) when $q \neq 1$. From studying the Wien approximation and formula (32) a similar behaviour for the energy density distribution (29) can be expected. In fact, a numerical analysis of $u_\tau(x)$ is shown in figure 2 for the same τ values as in the Wien distributions $u_\tau^{\mathrm{W}}(x)$ (figure 1). Both families of curves are in qualitative agreement and even in quantitative agreement except when x approaches 0 where the Wien law fails. Thus we arrive at the conclusion that the parameter q qualitatively acts on the energy distributions as the temperature parameter does in the 'classical' Planck law, but with the difference that any variation of q always produces a decrease in the total energy while the temperature may decrease or increase the total energy according to the temperature variation.

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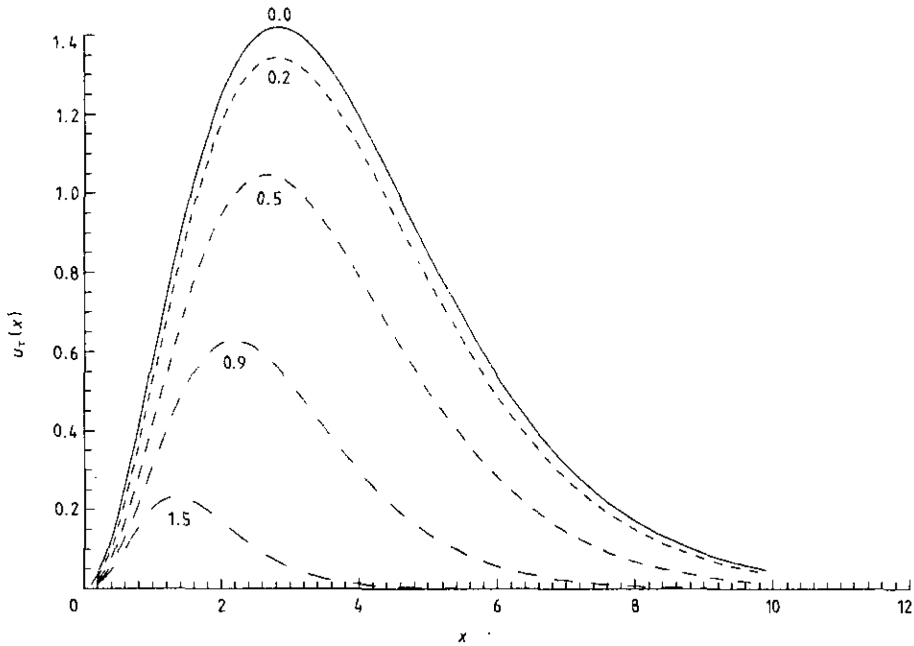

Figure 2. Numerical analysis of energy density curves $u_\tau(x)$ (30) for different values of the deformation parameter $\tau = \ln q$.

References

[1] Drinfeld V G 1986 *Quantum Groups Proc. Int. Congr. of Mathematics* (Berkeley, CA: MSRI) 798; 1988 *Sov. Math. Dokl.* **36** 212

[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247

[3] Manin Yu I 1988 *Quantum Groups and Non-Commutative Lie Algebras Preprint* University of Montreal CRM-1561

[4] Woronowicz S L 1987 *Commun. Math. Phys.* **111** 613

[5] Moore G, Seiberg N 1988 *Phys. Lett.* **212B** 451

[6] Alvarez-Gaumé L, Gómez C and Sierra G 1989 *Nucl. Phys.* B **319** 155

[7] Alvarez-Gaumé L, Gómez C, and Sierra G 1989 *Phys. Lett.* **220B** 142

[8] Alvarez-Gaumé L, Gómez C and Sierra G 1990 *Nucl. Phys.* B **330** 347

[9] Baxter R J 1982 *Exactly Solvable Models in Statistical Mechanics* (New York: Academic)

[10] Faddeev L D, Reshetikhin N and Takhtajan L 1987 *Quantization of Lie Groups and Lie Algebras Preprint* LOMI Leningrad

[11] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581

[12] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873

[13] Biedenharn L C 1989 Lecture given at the Arnold Sommerfeld Institute summer workshop on *Quantum Groups* (Federal Republic of Germany: Clausthal)

[14] Chainchian M and Kulish P 1990 *Phys. Lett.* **234B** 72

[15] Chainchian M, Kulish P and Lukierski J 1990 *Phys. Lett.* **234B** 401

[16] Floreani R, Spiridonov V P and Vinet L 1990 *Phys. Lett.* **242B** 383

[17] Soni S K 1990 *J. Phys. A: Math. Gen.* **23** L951

[18] Martín-Delgado M A *J. Phys. A: Math. Gen.* accepted for publication

[19] Yan H 1990 *J. Phys. A: Math. Gen.* **23** L1155

[20] Abe E 1980 *Hopf Algebras (Cambridge Tracts in Mathematics)* (Cambridge: Cambridge University Press)

[21] Kirillov A M and Reshetikhin M Yu 1988 *Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of Links Preprint* LOMI Leningrad E-9-88

[22] Galindo A and Pascual P 1978 *Mecánica Cuántica* (Madrid: ED Alhambra)